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# ASYMPTOTIC SOLUTIONS FOR NON-LINEAR SYSTEMS WITH HIGH DEGREES OF NON-LINEARITY†

I. V. ANDRIANOV

Dnepropetrovsk

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A method is proposed for the recurrent construction of the periodic solution of a substantially non-linear conservative system with a single degree of freedom which is close to a vibration impact system. It is assumed that the restoring force is a power function of the deflection. A quantity which is the inverse of this exponent is regarded as a small parameter. The method is based on the asymptotic representation (in a certain weak sense) of this non-linearity in powers of a small parameter) using normalization and Laplace transformation procedures. This approach leads to differential equations containing generalized δ-functions of the unknown variable and derivatives of these functions of as high an order as desired.

THE CONSTRUCTION of a sequential asymptotic procedure, based on a power expansion in $n^{-1}$, where $n$ is the degree of non-linearity, to some extent solves the problem of justifying the Π-method [1–3]. Here, results based on the Π-method are obtained as the zeroth approximation just like, for example, results based on the Van der Pol method serve as the zeroth approximation in the Krylov–Bogolyubov–Mitropol'skii averaging procedure.

As an example, we will consider the equation

$$x^{\cdot\cdot}+x^n=0,\quad n=2k+1,\quad k=1,2,\ldots$$

for which we will seek a single parameter family of periodic solutions which are skew-symmetric with respect to the origin of coordinates in the limit as $n\to\infty$.

Let us introduce the function $\xi=x/A$ ($A$ is the amplitude) for which the inequality $0\leqslant|\xi|\leqslant 1$ holds. Note that the function $\xi$ is continuous and periodic.

The initial equation can then be represented as follows:

$$\xi^{\cdot\cdot}+A^{n-1}\xi^n=0 \tag{1}$$

We will expand the function $\xi^n$ in series in $1/n$ as $n\to\infty$. In order to do this, we first transform the function

$$\varphi=\begin{cases}\xi^n, & 0\leqslant\xi\leqslant 1\\ 0, & \xi>1\end{cases}$$

using a Laplace transformation $\varphi(\xi)\to p^{-n-1}\gamma(n+1,p)$.

On expanding the incomplete gamma function $\gamma(n+1,p)$ in series in $1/n$ and, on carrying out the

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inverse transformation in a term-by-term manner (this procedure is justified in [4, 5], for example), we obtain

$$\varphi = \delta(\xi-1)(n+1)^{-1} - \delta(\xi-1)(n+1)^{-1}(n+2)^{-1} + \dots \tag{2}$$

where $\delta(\cdot)$ is the delta function.

We will now make the change of variable $t = \tau/\omega$ in Eq. (1).

On retaining just the principal term in the second sum and putting

$$\omega^2 = A^{n-1}/(n+1) \tag{3}$$

(since $0 \leq |\xi| \leq 1$), we have the equation

$$d^2\xi_0/d\tau^2 = -\delta(\xi_0 - 1) \tag{4}$$

for determining the periodic function $\xi_0$.

We will now consider the mathematical meaning of Eq. (4). On its right-hand side, there is a generalized function which is localized on the line $\xi_0 = 1$. This is a common object in the theory of generalized functions [6], and therefore none of the difficulties which occur in problems with impact interactions [7] arise here.

Integration of Eq. (4) taking account of the skew symmetry with respect to the origin of coordinates yields in the initial variables

$$x_0 = A\omega t \tag{5}$$

Expression (3), which can be treated as an amplitude–frequency dependence, and the solutions over a quarter of a period agree with those obtained by the Π-method [1–3].

We will now construct the subsequent approximations.

In order to do this, we will first represent $\xi$ in the form of a series

$$\xi = \xi_0 + \xi_1(n+2)^{-1} + \dots \tag{6}$$

On substituting series (6) into expression (2) and expanding the latter with respect to $(n+2)^{-1}$, we have

$$\begin{aligned} \delta[\xi_0 + \xi_1(n+2)^{-1} + \dots - 1] &= \delta(\xi_0 - 1) + \xi_1(n+2)^{-1}\delta'(\xi_0 - 1) + \dots \\ \delta'[\xi_0 + \xi_1(n+2)^{-1} + \dots - 1] &= \delta'(\xi_0 - 1) + \xi_1(n+2)^{-1}\delta''(\xi_0 - 1) + \dots \end{aligned} \tag{7}$$

Formulae (7) are obtained after a transition into the image space, expansion of the right-hand sides of the corresponding expressions in series with respect to $(n+2)^{-1}$ and then carrying out the inverse transformations. In addition, we introduce the expansion of $\omega$ in powers of $(n+2)^{-1}$

$$\omega = [(A^{n-1}/(n+1))]^{1/2}[1 + \omega_1(n+2)^{-1} + \dots] \tag{8}$$

After substituting relationships (6) and (8) into Eq. (1), making the change of variable $t = \tau/\omega$ and splitting with respect to $(n+2)^{-1}$, we obtain

$$d^2\xi_1/d\tau^2 = -[1 - \xi_1]\delta'(\xi_0 - 1) + 2\omega_1\delta(\xi_0 - 1) \tag{9}$$

The occurrence, on the right-hand side of (9), of a derivative of a δ-function leads to the build up of a higher-order singularity in the solution. In order to remove this singularity, we put

$$\xi_1(1) = 1 \tag{10}$$

Then

$$d^2\xi_1 / d\tau^2 = 2\omega_1\delta(\xi_0 - 1) \tag{11}$$

The solution for $\xi_1$ can be represented in the form $\xi_1 = \tau$, and we then find that $\omega_1 = -1/2$ from the boundary condition (10). The higher approximations are constructed in a similar manner although, of course, this is a fairly lengthy process.

We note that the smoothness of the solution when $\tau = 1$ is violated during the sequential asymptotic integration. In order to remove this difficulty, it is possible to up the preservation of asymptoticity, by taking account of terms of a higher order of smallness.

Equation (11) then takes the form

$$d^2\xi_1 / d\tau^2 = 2\omega_1\xi_0^n \tag{12}$$

The solution of Eq. (12) with boundary condition (10) is identical with the first approximation of the iteration procedure which has been previously suggested [1–3].

The formal asymptotic procedure is described above. Questions of convergence, estimates of accuracy, etc., have not been considered.

The approach proposed is a natural asymptotic method for solving differential equations containing terms of the form of $x^{1+\alpha}$, when $\alpha \to \infty$. A method has been developed in [8] for constructing the asymptotic form of similar equations when $\alpha$ is small. The existence of solutions when $\alpha \to 0$ and when $\alpha \to \infty$ allows one subsequently to use the apparatus of two-point Padé approximants [9] and to obtain a unique solution for any $\alpha$.

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